Russell's Law

An Ode to a Logical Theorem*

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I

Bertrand Russell wrote of the paradox that now bears his name in chapter X of *The Principles of Mathematics* (Cambridge University Press, 1903).[1] Russell's paradox concerns the set ("class") of sets that are not elements of themselves. As Russell was aware, there is a variant of this paradox that concerns properties (or concepts or propositional functions) instead of sets. We consider the property $\mathfrak{R}$ of being a property that is not a property of itself. Is $\mathfrak{R}$ a property of itself? If it is, then it is not a property of itself. Therefore $\mathfrak{R}$ is not a property of itself. But then $\mathfrak{R}$ is a property that is not a property of itself, and thus $\mathfrak{R}$ is a property of it. In that case, it is a property of itself. In appendix B, §500 (p. 527) of *Principles* Russell introduced a variant of

* I owe an enormous debt to the late Alonzo Church, who taught me (among many things) the ramified theory of types, Gödel's incompleteness theorems, and Tarski's theorem about truth. Any deficiencies in my knowledge of such matters are entirely my fault. I am also indebted to the outstanding textbook from which I learned classical first-order logic: *Logic: Techniques of Formal Reasoning* (Oxford University Press, 1964, 1992) by Gary Mar, the late Richard Montague, and my former teacher in set theory and meta-logic, the late Donald Kalish. The present essay is a result of the "diversion" I have found in the text's delightful T269 (p. 228). I am grateful to C. Anthony Anderson and Teresa Robertson for discussion of the topics of the present essay, and to my audience at UCLA in April 2014 for reactions. I am especially indebted to my teacher in introductory logic in 1971, Harry Deutsch, for extremely helpful e-discussion.

the paradox, one that concerns in lieu of Russell's set $\{y\colon y \text{ is a set } \& \ y \notin y\}$ and Russell's property $\mathfrak{R}$, a proposition about propositions. Where **M** is any property of propositions, the particular proposition that every proposition having **M** is true—in symbols, the proposition that $\forall q(\mathbf{M}q \rightarrow q)$—itself either has or lacks the invoked restricting property **M**. Let **W** be the property of being such a proposition that lacks its restricting property. Let p be the particular proposition that $\forall q(\mathbf{W}q \rightarrow q)$. It is deducible that p both has and lacks **W**.[2]

Whereas Russell's paradox and its variant concerning properties are obviously solved by the simple theory of types, the Appendix-B paradox is generally thought to be left unsolved by simple type theory.[3] Yet Russell says in *Principles* that "the close analogy of this contradiction with the one discussed in Chapter X strongly suggests that the two must have the same solution, or at least very similar solutions" (p. 528). He no sooner considers extending a version of the "theory of types" to propositions as a possible solution than he dismisses the idea as "harsh and artificial." The appendix, and therewith the book, closes one page later with a heroically candid admission of defeat:

> To sum up: it appears that the special contradiction of Chapter X is solved by the doctrine of types, but that there is at least one closely analogous contradiction which is probably not soluble by this doctrine. The totality of all logical objects, or of all propositions, involves, it would seem, a fundamental logical difficulty. What the complete solution of the difficulty may be, I have not succeeded in discovering; but as it affects the very foundations of reasoning, I earnestly commend the study of it to the attention of all students of logic (p. 528).

In setting out the Appendix-B paradox Russell invokes the notion of logical product (*every proposition having* **M** *is true)*. That notion is inessential. An exactly analogous paradox is obtained by replacing the proposition that $\forall q(\mathbf{M}q \rightarrow q)$ with the proposition that predicates a property Φ of **M**, for any property Φ of properties of propositions. Not being possessed by untrue propositions is one instance of such a property Φ but not otherwise remarkable. Let $\mathbf{W}_\Phi$ be the property of being a proposition *q* such that ∃**M**[(*q* = that Φ**M**) & ~**M***q*]. Let p_Φ be the singular proposition that Φ$\mathbf{W}_\Phi$. If p_Φ has $\mathbf{W}_\Phi$, then for some property **M** of propositions, p_Φ = that Φ**M**, and furthermore p_Φ lacks **M**. The property **M** is $\mathbf{W}_\Phi$ itself. It follows that p_Φ lacks $\mathbf{W}_\Phi$. But in that case ∃**M**[(p_Φ = that Φ**M**) & ~**M**p_Φ], and thus p_Φ has $\mathbf{W}_\Phi$ after all.

The Appendix-B paradox and a host of similar paradoxes thought unsolved by simple type theory are in fact partially solved in simple type theory without resorting either to set theory or to ramification in the usual sense. Even semantic paradoxes like that of the liar are partially solved by simple type theory without stratifying semantic terms like 'satisfies' and 'true'.[4] Indeed, these paradoxes are partially solved by classical first-order logic with its simple theory of types. (The sense in which the solution is partial will be explained below.)

I first pose a variant of the Appendix-B paradox that avoids quantification over propositions. Let us say that a proposition *q* is *monadic-predicative* if for some object *x* and some property F, *q* is the (or, at least, *a*) proposition to the effect that F*x*. Let us say further that a proposition *q predicates* a property F *of* an object *x* if *q* is the (or a) monadic-predicative proposition to the effect that F*x*.[5] We distinguish two kinds of monadic-predicative propositions. A proposition is a *kettle caller* if it is monadic-predicative and possesses the very property that it predicates. A proposition is a *stone caster* if it is monadic-predicative and not a kettle caller, i.e., it lacks the property that it predicates.[6] Every monadic-predicative proposition is either a kettle caller or a stone caster and no proposition is both. Monadic-predicative propositions are typically stone casters. The monadic-predicative proposition about Socrates that he is married is itself unmarried; thus it is a stone caster. By contrast, the monadic-predicative proposition that Fermat's Last Theorem is refutable is not only false but refutable; thus it is a kettle caller. Let *r* be the stone caster that Socrates is married, and let $\wp$ be the true monadic-predicative proposition that *r* is a stone caster. What of $\wp$ itself? Suppose that $\wp$ is a stone caster. It then lacks the property it predicates, which is that of being a stone caster. Therefore $\wp$ is not a stone caster; it is a kettle caller. Since it then possesses the very property that it predicates, and that is the property of being a stone caster, $\wp$ is a stone caster after all. But no proposition can be both a kettle caller and a stone caster.

Both the stone-caster paradox and the Appendix-B paradox are variants of Russell's paradox. The stone-caster paradox is more like the Appendix-B paradox, however, in that it invokes a property of propositions—that of being a stone caster (in lieu of Russell's **W**)—and a paradox-producing proposition $\wp$ about this property (in lieu of Russell's *p*).

II

Some very significant results in philosophical logic are based to a very large extent on a theorem of first-order logic, which I call 'Russell's law':

$$\sim\exists x \forall y[\mathrm{R}(xy) \leftrightarrow \sim\mathrm{R}(yy)] .$$

An *interpretation* **I** of Russell's law consists of a particular universe **U** over which the variables '*x*' and '*y*' range coupled with the assignment of a particular binary relation **R** defined over **U** as the semantic extension of the dyadic predicate 'R'. (If **U** is a set, the interpretation is a *model* of Russell's law.) Russell's law is true no matter the interpretation. Russell illustrated the law by means of the example of the village barber who shaves all and only those villagers who do not shave themselves. The riddle: Does the barber shave himself? In interpretation $\mathbf{I}_{\mathbf{VB}}$ we let **U** be the set of villagers and we let **R** be the following relation: *x* shaves *y*.[7] Russell's law states on $\mathbf{I_{VB}}$ that there is no such barber. The supposition that there is such a barber is logically inconsistent, a violation of a logical edict. Alternative interpretations determine alternative instances. Many of these are quite remarkable; all of them are un-evadable. On exactly the same purely logical grounds, there is no one who admires all and only those who do not admire themselves, no list of all and only those lists that do not list themselves, and no property $\Re$ had by all and only those properties that are not properties of themselves. Each is a truth of first-order logic, an interpretation instance of Russell's law.[8]

If the universe **U** over which the '*x*' and '*y*' range is restricted, the variables '*x*' and *y*' may be interpreted instead as having unrestricted range, while representing the intended restriction by means of a monadic predicate 'U'. The dyadic predicate 'R' is a predicate-constant that may be interpreted as standing for any suitable binary relation **R** between elements of **U**. To obtain an actual principle (a single, true proposition), both 'U' and 'R' should be regarded as predicate-variables bound by initial universal quantifiers:

$$\forall U \forall R {\sim} \exists x[\ Ux \ \&\ \forall y(\ Uy \rightarrow [R(xy) \leftrightarrow {\sim}R(yy)]\ )\ ]$$

This theorem of second-order logic is perhaps what is most aptly called 'Russell's law'.

Its breadth is genuinely remarkable. Besides pointing the way to solutions to a number of paradoxes Russell's law is also the basis for significant results in logic and the foundations of mathematics.

A paradox, in the sense in which I am using the term, inevitably involves logic in that it consists in the deduction of a contradiction. The deduction is a genuine paradox (sometimes called an *antinomy*) only insofar as each of the premises essentially involved in the deduction seems true and each of the inferences seems valid. My claim is that for a host of paradoxes

Russell's law is not merely involved in the sense in which logic is inevitably involved in deduction; Russell's law points the way to a solution. A solution to a paradox must identify the faulty principle(s) and/or fallacious inference(s) that are essentially involved in the derivation of the contradiction. To do only this is to provide only a partial solution. As Aristotle insightfully noted, "We must, however, not only state the true view, but also give the explanation for the false one, since that promotes confidence. For when we have a clear and good account of why a false view appears true, that makes us more confident of the true view" (*Nichomachean Ethics* 1154a).[9] A complete solution to a paradox replaces the flawed principles and/or inference patterns with corrected, weakened variants that accomplish much or (preferably) all of what was wanted without also issuing a contradiction. Russell's law does not by itself completely solve any paradox. However, suitably interpreted, Russell's law points the way to genuine resolution for each of a host of paradoxes, by exposing the error of an otherwise compelling principle involved in deducing the contradiction.

To illustrate, in interpretation $\mathbf{I_{RP}}$ we let **U** be the universe of all sets and let **R** be the inverse $\ni$ of $\in$ (i.e., the relation: $\lambda xy[y \in x]$). On $\mathbf{I_{RP}}$ Russell's law states that there is no such set as the putative set of sets that are not elements of themselves. It follows that unrestricted comprehension of naïve set theory is incorrect:

$$\exists x \forall y (y \in x \leftrightarrow \phi_y)$$

This is a schema. It represents not a single principle but each of the infinitely many instances obtained by putting in an actual formula without 'x' free for the symbol 'ϕ_y'. The particular instance obtained by letting ϕ_y be '$y \notin y$' is precisely what Russell's law on $\mathbf{I_2}$ denies. That instance of unrestricted set-theoretic comprehension is logically inconsistent.[10]

Unrestricted set-theoretic comprehension may be taken to be the second-order rendering: $\forall F \exists x \forall y(y \in x \leftrightarrow Fy)$. Unlike the schema this is a single principle, a true-or-false proposition. Instantiating the initial universal quantifier to '$\lambda z[z \notin z]$' yields a disproof by providing a counter-instance. The inferred consequence is first-order-logically inconsistent.

It is frequently said in this connection that the class of all sets that are not elements of themselves is too large to be a set.[11] As an explanation this misses the point. It is like saying that the village barber does not exist because there are too many villagers who do not shave themselves for a single barber to manage, or more dramatically, that the would-be barber is excessively obese. Russell's putative set does not exist for the same reason the village barber does not exist: the supposition that there is such a thing is inconsistent, in first-order logic in fact. Even if, *per impossibile*, the sets that are not elements of themselves were relatively few in number, by Russell's law there could not be a set of all and only them.

Russell's law does not formally preclude there being something whose elements are all and only sets that are not elements of themselves. It requires that nothing with that membership be a set. Also, no such thing can be among its own elements since its membership is restricted to sets. Contemporary class theory, which originated with John von Neumann, posits entities that have elements just as sets do but are not sets. Taking '∅' as a primitive symbol for the empty set, one might define a *class* (as contrasted with a set) as anything that is either ∅ or has elements. All sets are classes, but according to von Neumann's theory the converse does not obtain. A *proper class* is a class that is not also a set. In particular, contemporary class theory posits a class, *R*, whose elements are precisely all and only sets that are not elements of themselves. Insofar as sets are distinguished from classes, the following must be distinguished conceptually: (*i*) the Russell set, i.e., the putative set of sets that are not elements of themselves; (*ii*) the class of classes that are not elements of themselves; and (*iii*) the class *R* of sets that are not elements of themselves. Russell's law precludes (*i*) and (*ii*) but not (*iii*).

In their *Foundations of Set Theory* (Amsterdam: Elsevier, 1958), Abraham A. Fraenkel and Yehoshua Bar-Hillel write, "It was von Neumann's daring idea that it is not the existence of the overcomprehensive sets as such which leads to contradictions, but *their being taken as members of other sets* (elementhood)" (p. 97). In the second edition (1973), Fraenkel, Bar-Hillel, and Azriel Levy say something very similar:

> Von Neumann regards as the main idea of his set theory the discovery that the antinomies do not arise from the mere existence of very comprehensive sets, but from their elementhood, i.e., from their being able to be members of other sets (pp. 135-136).[12]

According to von Neumann-Bernays-Gödel (NBG) set theory, a set is a class that is an element of some class or other, and a proper class is a class that is not an element of any class.[13]

These statements require a number of clarifications and corrections. In the first place, a set's capacity for being a member of classes—its elementhood—does not of itself lead to contradiction. If it did, the only sets would be those (including ∅) whose elements are exclusively *ur*-elements. Second, the remarks conflate a set's membership with its existence. Though intimately related, these properties must be sharply distinguished, especially when distinguishing either of them from a third property (elementhood). What Fraenkel, *et al*. evidently mean would be better put by saying that the central idea underwriting von Neumann's class theory is that it is not the overcomprehensive membership, *per se*, that logically precludes a paradoxical set's existence, but the *combination* (conjunction) of the very comprehensive membership together with elementhood. Assuming there is such a thing as a von Neumann proper class—a class that is not itself a class-member—the von Neumann class of non-self-membered sets has the membership of Russell's putative set without the elementhood. Third, even as restated the claim of logical incompatibility is questionable, as is the analysis of sethood as classhood-*cum*-elementhood. If there is such a class as R, why not also its singleton $\{R\}$? One might interpret von Neumann and company instead as meaning by '$x \in y$' not merely that x is an element (member) of y but that x is both an element of *y and a set* (or an *ur*-element).[14] What gives rise to contradiction and thereby precludes Russell's putative set is the combination of R's membership together with the property of *being a set*. By Russell's law—by first-order logic alone—nothing has that combination of properties.[15]

Not least, contrary to the quoted remarks the "mere existence" (by contrast with its presumed elementhood) of a genuine set whose elements are all and only sets that are not elements of themselves does lead inexorably to a contradiction, thereby giving rise to an antinomy. Russell's law—a logically true, general negative-existential—is, as far as it goes, a completely correct response. On pain of contradiction, there is no such set. Indeed, by Russell's law there is nothing—no set, no class, no thing-with-elements—whose elements are all and only those things that are not elements of themselves.

III

Russell's law might be more aptly called 'Cantor's law', since it is also the basis for Georg Cantor's method of diagonalization, which led to the discovery of Russell's paradox. To illustrate, let E be an arbitrary enumeration of denumerably many infinite sequences of the integers 1 and 0 (alternatively, of the digits '1' and '0'). Diagonalization extracts from E an infinite sequence of 1's and 0's that is not among those enumerated, by switching the first element of the first sequence of E with the other digit (either 1 to 0 or 0 to 1), and doing the same with the second element of the second sequence of E, the third element of the third sequence of E, and so on. In interpretation $\mathbf{I_D}$ we let the universe **U** be the set of positive integers {1, 2, …}, and we let **R** be: $[E(x)](y) = 1$, i.e., sequence x of E has a 1 rather than a 0 at position y. On $\mathbf{I_D}$ Russell's law states that the particular diagonal sequence of 1's and 0's constructed in this manner is not among those enumerated by E. It follows that the set of all sequences of 1's and 0's is nondenumerable (uncountable), since as a matter of first-order logic, any enumeration of sequences of 1's and 0's inevitably omits at least one such sequence.

Results obtained by diagonalization are thus traceable to Russell's law. One obvious example is the nondenumerability of the reals. A related interpretation of Russell's law yields Cantor's theorem about power sets. In $\mathbf{I_{PS}}$ we let the universe **U** be an arbitrary set. Let f be any function from **U** to subsets of **U**, i.e., into **U**'s power set, $\mathbb{P}(\mathbf{U})$. Now let **R** be the binary relation: $y \in f(x)$. On $\mathbf{I_{PS}}$ Russell's law states that f does not assign to any element of **U** the particular set $\{y: y \in \mathbf{U} \;\&\; y \notin f(y)\}$, i.e., the set of elements of **U** that are not elements of the very subset that f assigns them. Assuming that the unassigned subset exists—it does if Separation is true—f is not onto $\mathbb{P}(\mathbf{U})$; it falls short of assigning all subsets of **U** to elements of **U**.[16] Since Russell's law on $\mathbf{I_{PS}}$ holds for any set **U**, and for any function f from **U** into its power set, any function from any set into its power set inevitably fails to assign the specified subset to any of the set's elements.[17] It follows, assuming Separation, that there is no bijection between an arbitrary set and its power set, so that the cardinality of the latter invariably exceeds that of the former.

Combining aspects of the applications of Russell's law to Russell's paradox and to Cantor's theorem, in $\mathbf{I_V}$ we let **U** be an arbitrary set and let **R** be $\ni$. On $\mathbf{I_V}$ the law states that the Russell subset of **U**, *viz.*, $\{y: y \in \mathbf{U} \;\&\; y \notin y\}$, is no element of **U**. Since this result holds for any set whatsoever, the Russell subset of any set is not among that set's elements. Assuming

Separation, so that the Russell subset of any set exists, it follows that there is no set of all sets (and that the von Neumann universe **V** of hereditary well-founded sets is not a set).[18]

Russell's law ultimately precludes the possibility of there being an open formula, *Satisfies*(*x*, *y*), with '*x*' and '*y*' its only free variables, of a language of Peano arithmetic that is semantically satisfied in the language (in the sense of Alfred Tarski) by all and only those pairs of natural numbers <*n*, *m*> such that *n* satisfies in the language the formula whose Gödel number is *m*. For according to Russell's law there can be no open formula of a language for Peano arithmetic, *Non-self-satisfying*(*x*) with '*x*' its only free variable, that is satisfied in the language by all and only the Gödel numbers of those formulae of the language that are not satisfied by their own Gödel numbers. In $\mathbf{I_T}$ we let **U** be the set of Gödel numbers of open formulae of a particular language of Peano arithmetic and let **R** be the relation: the formula whose Gödel number is *x* is satisfied in the language by *y*. If there were such a formula as *Satisfies*(*x*, *y*), then *Non-Self-Satisfying*(*x*) would be definable as ⌜~*Satisfies*(*x*, *x*)⌝. It follows from the nonexistence of *Satisfies*(*x*, *y*) that there is also no truth formula, *True*(*x*), of the language of Peano arithmetic, i.e., no formula satisfied in the language by the Gödel numbers of all and only the language's true sentences. For if there were, from it one could construct the mythical formula *Satisfies*(*x*, *y*).[19]

Although there is no truth formula of a language of Peano arithmetic, as Gödel showed there is a provability formula, i.e., an open formula, *Provable*(*x*), of the language of Peano arithmetic that is satisfied in the language by the Gödel numbers of all and only the theorems of Peano arithmetic. It follows that if Peano arithmetic proves only true sentences, then there are some true sentences of the language the Gödel numbers of which satisfy ⌜~*Provable*(*x*)⌝, and are therefore not provable in Peano arithmetic. Otherwise *Provable*(*x*) would be the mythical formula *True*(*x*) that Russell's law excludes. If Peano arithmetic proves only true sentences, then there are some sentences that it does not decide. This result is strictly weaker than Gödel's first incompleteness theorem, but it is an extremely important result.[20]

IV

The nonexistence of *Non-self-satisfying*(*x*) leads ultimately to rather surprising results in the philosophy of language and the metaphysics of semantics. In 1908 Kurt Grelling proposed the adjectives 'autological' and 'heterological' to be added to English. Let us give the name 'Grenglish' to the expanded language that results by adding these terms as well as the name 'Grenglish' itself to English, or to a suitable fragment thereof.[21] (It is to be presumed that the language in which the present essay is written is neither exactly English nor exactly Grenglish, but a suitable metalanguage for each.) The adjective 'autological' is defined so that it semantically applies in Grenglish to any adjective of Grenglish that semantically applies in Grenglish to itself (e.g., 'adjectival'); 'heterological' is defined so that it applies in Grenglish to any adjective of Grenglish that does not apply in Grenglish to itself (e.g., 'adverbial'). To dispel misguided attempts at solution, we stipulate further that neither 'autological' nor 'heterological' is in any way indexical or context-sensitive, and that both are monadic so that neither of the predicates 'is autological' and 'is heterological' conceals any hidden argument places. Thus with respect to every context 'heterological' means the following in Grenglish: *adjectival and non-self-applicable in Grenglish*.[22] Grelling's paradox arises as follows: If 'heterological' applies in Grenglish to itself, then it satisfies its own definition, so that it does not apply in Grenglish to itself. Therefore, 'heterological' does not apply in Grenglish to itself. Since it is thus an adjective that does not apply in Grenglish to itself, it satisfies the definition of 'heterological'. In that case the adjective 'heterological' applies to it in Grenglish. Therefore, 'heterological' applies in Grenglish to itself after all.

There is an analogous paradox that avoids the term 'heterological' altogether.[23] Consider the English noun phrase

h: general phrase that does not apply in English to itself .

(We stipulate that *h* is monadic and non-indexical.) This phrase applies in English to the phrase 'general phrase that consists of at least eight words', since the latter phrase consists of exactly seven words. By the familiar argument *h* applies in English to itself iff it does not.

Russell's law once again points to a way out of our conundrum. In $\mathbf{I_{het}}$ we let **U** be the set of (monadic and non-indexical) adjectives of Grenglish and let **R** be the relation: *x* applies in Grenglish to *y*. On $\mathbf{I_{het}}$ Russell's law states, on pain of contradiction, that there is no (monadic, non-indexical) adjective of Grenglish that applies in Grenglish to all and only those (such) adjectives of Grenglish that do not apply in Grenglish to themselves. Since there is no such adjective, in particular the Grenglish word 'heterological' is not such an adjective. For that matter, even the adjective phrase 'non-self-applicable in Grenglish' (assuming it is monadic and non-indexical) does not apply in Grenglish to all and only those adjective phrases of Grenglish that do not apply in Grenglish to themselves. Yet 'heterological' was defined to apply in Grenglish to all and only those adjectives of Grenglish that do not apply in Grenglish to themselves. Russell's law entails that the attempt to construct a term of Grenglish with the intended application condition "did not take."[24]

How exactly does the definition misfire? On this point my thoughts are less definite. The definition of 'heterological' represents an attempt to stipulate or fix semantic-application conditions in Grenglish for 'heterological'. Yet the definition evidently also presupposes that every adjective of Grenglish has its semantic-application conditions already fixed in advance. The definition's objective clashes with its presuppositions. Perhaps the definition thus suffers from a kind of circularity. On the other hand, the alternative version of Grelling's paradox invoking the phrase *h* demonstrates that the primary deficiency applies as much to the definiens as to the definiendum, and indeed is not restricted to definitions.[25] That deficiency is not that there is no relevant concept of a non-self-applicable English general term. The concept exists, and is perfectly well expressible in any suitable metalanguage. Indeed it is expressed by the natural translation of *h*. Whatever the precise reason, this much is certain: The adjective 'heterological' does not apply in Grenglish in the intended manner; furthermore the phrase *h* does not apply in English in the intended manner. By Russell's law there can be no Grenglish adjective that does and there can be no English phrase that does.

Not all of the paradoxes of naïve set theory and naïve semantics reduce to a single simple law of logic. But Russell's and Grelling's paradoxes—arguably the simplest paradoxes of each kind—do so reduce. The mere fact that Russell's law points the way to resolving both Russell's paradox and Grelling's in one fell swoop discredits the claim that the resemblance between the two kinds of paradox is only superficial. In both cases, Russell's law requires that a certain naïve comprehension principle be restricted. In the case of naïve set theory, the guilty principle is unrestricted set-theoretic comprehension. Quine argued in effect that in the case of Grelling the naïve principle analogous to unrestricted set-theoretic comprehension is a semantic schema for general terms along the lines of the following:

GT: $\mathbf{Q}(\tau)$ applies in $\mathcal{L}$ to something if and only if it is (a) τ , where '$\mathscr{L}$' is to be replaced by the name of the language of the schema's instances, τ is to be replaced by a suitable general term (whether a word or a phrase) of that language, and $\mathbf{Q}(\tau)$ is to be replaced by a quotation (or other suitable canonical designator) of that term.[26] Correct instances of *GT* include ' 'disagreeable' applies in English to something iff it is disagreeable ' and ' 'brown-eyed girl' applies in English to something iff it is a brown-eyed girl '. Incorrect instances include

GT_{het}: 'heterological' applies in Grenglish to something iff it is heterological

GT_h: *h* applies in English to something iff it is a general phrase that does not apply in English to itself.

These instances are easily seen to be effectively inconsistent. One need only instantiate the universal quantifier in GT_{het} to 'heterological' itself, and the universal quantifier in GT_h to *h* itself. But to place the blame for the contradictions in the Grelling paradoxes entirely on *GT*, as Quine does, is to convict the henchman while allowing the kingpin to walk. To begin with, *GT* is not a single principle of English semantics. It is a schema that represents infinitely many semantic principles, one for each English general term. One can glean from the work of Tarski that instances of *GT* involving non-idiom phrases are not axiomatic. Rather they are derived from more fundamental truths of semantics concerning the individual words that make up the phrase and concerning the manner of composition of those words. The phrase 'brown-eyed girl' applies in English to all and only brown-eyed girls because the words 'brown', 'eye', and 'girl' mean what they do in English, and because of how a string consisting of an arbitrary adjective followed by an arbitrary common noun typically works in English, etc. But this makes the misfiring of GT_h all the more curious. The English word 'phrase' is a term for a phrase. The English word 'not' means negation. The English word 'English' designates English. The English phrase 'applies in … to' applies to all and only those triples $<\tau, \mathcal{L}, x>$ such that the general term τ applies in the language $\mathcal{L}$ to the object *x*. The expressions that make up *h* are combined grammatically. The phrase is well-formed; all of its ducks are in a row. Furthermore, the word-by-word "translation" of *h* into any suitable language applies to exactly those English phrases that do not apply in English to themselves Yet Russell's law reveals that GT_h must be incorrect. It is logically impossible for there to be a (monadic, non-indexical) English phrase that applies to exactly those English phrases that do not apply to themselves.

P*ace* Quine, the failure of *GT* traces further back, to the semantic rules governing compositionality (i.e., compounding). Typically, a grammatically well-formed string consisting of two general terms applies in English to all and only those things to which both of the general terms apply in English. This goes also for relevantly similar grammatically well-formed constructions (e.g., 'English general phrase that consists of at least eight words'). But *h* is a rare exception.[27] Some instances of *GT* are incorrect. GT_h is one.

Unrestricted compositionality is incorrect. Quite possibly, the semantic axioms governing compositionality in English, and in many other languages, are merely rules of thumb, with little recourse but to begin with the qualifier 'If not for every instance then to the fullest extent that logic allows, …'. What then of those general phrases like *h* that logically cannot but violate the English rule of thumb? Perhaps there are no English semantic rules that assign content to those rare beasts. If there are not, then *h* does not express anything at all in English. And in that case, *h* is—as may be said in the original language of the present essay but not in English—a general phrase that does not apply in English to itself. It will not follow that *h* applies in English to itself after all. What does follow is that the original language of the present essay is (as I cautioned above) not English but a suitable metalanguage.[28]

One is tempted to argue, analogously to the case of Peano arithmetic, that there is no monadic expression of English that applies in English (non-indexically) to all and only the true English sentences. (See note 23 above.) If this is so, then even the English phrase 'sentence that is true in English' does not so apply. This would be nothing short of bizarre. The English phrase 'sentence that is true in French', by comparison, surely applies in English to all and only the true French sentences. What goes wrong when 'French' is replaced by 'English'? Once unrestricted compositionality of naïve semantics is called into question, the bizarre is already at hand. What can be safely inferred from these considerations is that some instances of Tarski's *T*-schema for natural language fail, presumably because of a failure of unrestricted compositionality. In particular, if '*L*' names the English liar sentence '*L* is not true in English', then the corresponding *T*-sentence

'*L* is not true in English' is true in English if and only if *L* is not true in English

is not itself true in English.[29] If it were, the English liar sentence would be both true in English and not. What, if anything, the English liar sentence means in English is anything but obvious.

V

We return now to the Appendix-B and stone-caster paradoxes. In interpretation $\mathbf{I}_{\mathbf{AppB}}$ we let **U** be the class of monadic-predicative propositions that predicate Φ of some property **M** or other of propositions. To fix **R** we define a partial function M_Φ that assigns to each element of **U** the property of which it predicates Φ:

$M_\Phi(q) =_{df}$ the property **M** such that q predicates Φ of **M**.

As Russell notes (in effect), there is no more than one such property **M** for any such proposition q. Let **R** be the relation: $M_\Phi(x)$ is had by y, i.e., y has $M_\Phi(x)$. On $\mathbf{I_{AppB}}$ Russell's law states that there is no monadic-predicative proposition p_Φ that predicates Φ of the property $\mathbf{W}_\Phi$ of being a monadic-predicative proposition that lacks the very property of which it predicates Φ. Russell's law thus blocks Russell's Appendix-B paradox by exposing the nonexistence of the putative proposition p_Φ. The supposition that there is such a proposition is inconsistent.

Russell's law applies to the stone-caster paradox in an exactly analogous manner. In $\mathbf{I_{SC}}$ we let **U** be the class of monadic-predicative propositions, and let **R** be the relation: x predicates a property (some property or other) that y possesses. On $\mathbf{I_{SC}}$ Russell's law yields that there is no monadic-predicative proposition that predicates the property of being a stone caster. In particular, although r, the monadic-predicative proposition that Socrates is married, itself lacks the property of being married, there is no monadic-predicative proposition that predicates being a stone caster of r. There is no truth that r is a stone caster. Russell's law forbids.

I do not here provide a complete solution to the stone-caster paradox, which must both correct the error in the derivation of the contradiction and provide an explanation for our having committed the error in the first place. I do, however, submit that the following partially restricted comprehension schema of naïve proposition theory be seen as the relevant analog of the unrestricted comprehension schema of naïve set theory:

PC: $\forall x \exists q[q$ is monadic-predicative & q predicates τ of $x]$,

where τ is the gerund form of any monadic general term of English of a particular modestly restricted class (excluding at least the phrase 'property that does not possess itself'). This partially restricted schema may be seen as based upon an extremely plausible second-order principle: $\forall F \forall x \exists q$[$q$ is monadic-predicative & q predicates F of x]. The idea is that for every suitable well-formed gerund phrase τ of (any expansion of) English and for every object x, there is a monadic-predicative proposition that predicates the property designated by τ (assuming there is one) to x. The paradox demonstrates that the instance obtained by putting 'being a stone caster' for τ fails. In rejecting *PC*, I leave open the question of whether there are such properties as stone-casting. Provided that $\exists$F(F = being a stone caster), the second-order principle of propositional comprehension is false. Even if there is such a property as stone-casting, then even though *r* possesses it, there is no proposition that predicates it of *r*.

VI

By the time he wrote "Mathematical Logic as Based on the Theory of Types," Russell advocated an extremely elegant solution by what is now known as the ramified theory of types.[30] The central idea is that such things as propositions and attributes (I do not here distinguish between attributes and propositional functions) come in different levels ("orders"), with higher level propositions and attributes introduced ("defined") in terms of lower-level propositions and attributes. On ramified type theory, there is no unleveled proposition that Plato endorsed every proposition that Socrates asserted. Instead there is a level-2 proposition that Plato endorsed every level-1 proposition that Socrates asserted. On one formulation (there are alternatives), there is also a level-3 proposition, but no level-1 or level-2 proposition, that Aristotle doubted every level-1 and every level-2 proposition that Plato believed, and so on.

To take one example, ramified type theory solves Grelling's paradox by stratifying the property of heterologicality into a hierarchy of heterologicalities of different levels, and disambiguating the paradoxical adjective 'heterological' accordingly. We construct a language

Grenglish^R by adding to English (without the adjectives 'autological' and 'heterological') an infinite totality of adjectives: 'heterological^1', 'heterological^2', and so on. These adjectives are governed by the following definitions, one for each natural number $n \geq 0$:

> A Grenglish^R general term or monadic predicate τ is *heterological*$^{n+1}$, by definition, if and only if τ expresses as its semantic content in Grenglish^R a property of level lower than or equal to level n that is not possessed by τ itself.[31]

Grenglish^R replaces Grenglish as the operative language. For each n = 1, 2, …, heterologicality^{n+1} is the level-(n+1) property of being a general term that expresses in Grenglish^R a level-m property, $m \leq n$, that the term itself lacks. Any Grenglish^R general term that is heterological^m is also heterological^{m+1}, heterological^{m+2}, etc. Since a Grenglish^R general term is heterological^{n+1} only if the property it expresses is of level n or lower, and heterologicality^{n+1} is of level n+1, the adjective ⌜heterological^{n+1}⌝ lacks the very property that it expresses in Grenglish^R. Therefore ⌜heterological^{n+1}⌝ is heterological^{n+2}. Since heterologicality^m entails heterologicality^{m+1}, ⌜heterological^{n+1}⌝ is not heterological^{m+1} for any $m \leq n$, and is heterological^{m+1} for any $m > n$. The threat of contradiction is artfully fended off.

Ramified type theory provides an equally elegant solution to the stone-caster paradox. There are level-2 properties of kettle-calling^2 and stone-casting^2, level-3 properties of kettle-calling^3 and stone-casting^3, and so on. For n = 1, 2, …, a proposition q^n is a *kettle-caller*$^{n+1}$ if q^n is a monadic-predicative proposition of level n or lower and possesses the very property of that same level that it predicates. Analogously, q^n is a *stone-caster*$^{n+1}$ if q^n is a monadic-predicative proposition of level n or lower and is not a kettle-caller^{n+1}.[32] Let us assume that being married is level-1. The proposition that Socrates is married, r, is then a stone-caster^2 (and a stone-caster^{3}, etc.). For each $n = 0, 1, 2, \ldots$, let $\wp^{n+1}$ be the level-(n+1) monadic-predicative proposition that r is a stone-caster^{n+1}. (See again note 32.) The proposition $\wp^{n+1}$ itself is not of level n or lower, and is therefore not a stone-caster^{n+1}. Since it thereby lacks the property that it predicates, $\wp^{n+1}$ is a stone-caster^{n+2}. More generally, $\wp^{n+1}$ is not a stone-caster^{m+1} for any $m \leq n$, and is a stone-caster^{m+1} for any $m > n$. No contradiction looms.[33]

VII

On offer are two separate answers to the paradoxes: the partial solution by means of Russell's law, and Russell's own solution by means of ramified type theory. I have argued that the partial solution by means of Russell's law is un-evadable. However, the two answers may be viewed instead as complementary. The question is not which of the two to embrace, but whether the solution by means of ramified type theory should be accepted as completing the partial solution by means of Russell's law.

Two components of the ramified-type-theory solution should be distinguished. First, there is the stratification: the leveled propositions, properties, and functions. In addition to and separate from stratification is the simultaneous repudiation of simple type theory's unleveled propositions, properties, and functions. The ramified type theory's levels are not simply adjoined to simple types; ramified types *replace* their simple counterparts, which are regarded as logical nonsense. While it welcomes quantification over all propositions or properties of any particular level, Russell's solution prohibits quantification over all propositions or properties without regard to level. One may say that Socrates asserted^{m} a level-n proposition, where $m > n$, but to utter 'Socrates asserted at some level or other a proposition of some level or other' is to assert nothing of level 1, nothing of level 2, and so on. On Russell's account, it is therefore to assert nothing at all—despite the fact that it feels for all the world like truthful assertion. The unleveled claims that ⌜heterological^{n+1}⌝ is heterological *simpliciter* (or heterological^{m+1} *for some natural number m or other*), and that *r* is a stone caster *simpliciter*, are eschewed as logical nonsense—despite the fact that they feel for all the world like truths.

Stratification *per se* is deep and insightful. The levels of ramified type theory can scarcely be doubted. Surely some propositions and their predicative components do indeed come in levels of exactly the sort that the theory posits. But it seems also obvious, perhaps equally so, that there is in addition the general, unlevelled concept of a proposition or property *of some level or other*. It apparently makes perfectly acceptable sense to say that Napoleon has all the properties of every level of a great general, and it apparently makes perfectly acceptable sense to say that there is a proposition of some level or other that all men are created equal.

Ramified type theory's hierarchies of leveled heterologicalities and leveled stone-castings, is, at bottom, a change of subject. To illustrate this point we may expand Grenglish^{R} into a language Grenglish^{R+} by adding one more adjective: A Grenglish^{R+} general term τ is *heterological*$^{\exists}$, by definition, if and only if τ expresses as its semantic content in Grenglish^{R+} a property *of some level or other* not possessed by τ itself. Does 'heterological$^{\exists}$' apply in Grenglish^{R+} to itself? Each of the two incompatible answers to our new question leads to the other. The original question was whether 'heterological' is heterological (unleveled) in Grenglish. Likewise, the question was not whether $\wp^{n+1}$ is itself a stone-caster^{n+1}. The question was whether $\wp$ is itself a stone caster. Paradox is not solved by changing the subject.[34]

It is not the stratification *per se* that blocks the derivation of the contradiction on Russell's account. It is the prohibition on unleveled propositions and properties. Russell's attitude toward the paradoxical questions themselves was that they are ill-formed nonsense. But are they? The word 'heterological' is a part of Grenglish. Can it be sincerely denied, on reflection, that it makes no sense to say that the very word either applies (without level) in Grenglish to itself or it does not? If this does make sense, then a genuine solution to the paradox must specify which disjunct is correct.[35]

As the philosophical basis for the rejection of simple type theory's unleveled propositions and properties Russell offered his principle that "no totality can contain members defined in terms of itself"—*the vicious-circle principle* (*op. cit*., at p. 237; p. 75 of Marsh). In their monumental *Principia Mathematica* Whitehead and Russell elaborate:

> An analysis of the paradoxes to be avoided shows that they all result from a kind of vicious circle. The vicious circles in question arise from supposing that a collection of objects may contain members which can only be defined by means of the collection as a whole. Thus, … given any set of objects that, if we suppose the set to have a total, it will contain members which presuppose this total, then such a set cannot have a total. By saying that a set "has no total," we mean, primarily, that no significant statement can be made about "all its members." … In such cases, it is necessary to break up our set into smaller sets, each of which is capable of a total. This is what the [ramified] theory of types aims at effecting.
>
> The principle which enables us to avoid illegitimate totalities may be stated as follows: "Whatever involves *all* of a collection must not be one of the collection"; or, conversely: "If, provided a certain collection had a total, it would have members only definable in terms of that total, then the said collection has no total." We shall call this the "vicious-circle principle," because it enables us to avoid the vicious circles involved in the assumption of illegitimate totalities (*Principia Mathematica*, *Volume I*, Cambridge University Press, 1910, 1927, at p. 37).

In "Mathematical Logic as Based on the Theory of Types" Russell infers from the vicious-circle principle that "whatever contains an apparent [i.e., bound] variable must not be a possible value of that variable" (p. 237). In short, according to Russell the extension or designatum of a well-formed expression that contains a bound variable cannot be an element of the universe over which that variable ranges.

The vicious-circle principle, as understood by Russell, is susceptible to the most reasonable of doubt. As many have noted, designating definite descriptions, like 'Russell's favorite barber' (and including set-theoretic descriptions of the form ⌜{α: ...α...}⌝), at least as they are conceived by Mill, Frege, Church and numerous other theorists, do not conform with the restriction that Russell infers from his vicious-circle principle. Russell's formulation of the principle employs the questionable locution of "defining" a non-linguistic, logico-mathematical object, perhaps in the sense of *introducing* or *postulating* the object by defining a term to designate it.[36] Plainly, circular definition is defective. At the very least it is treif; one might even say that it is vicious. I suggested above that the introductory definition of 'heterological' for Grenglish perhaps suffers a kind of circularity, in that it represents an attempt to stipulate the semantic-application conditions in Grenglish for 'heterological' at the same time that it presupposes that the adjectives of Grenglish, including 'heterological', already have their semantic-application conditions fixed. It is by no means obvious, however, that the definition of '*w*' in appendix B, or of 'stone caster', is subject to an analogous sort of clash, or indeed to any genuinely vicious circularity. The only definite presupposition failures are the presuppositions that the paradoxical proposition exists (and that '*p*' or '℘' designate).

By contrast, an alternative "vicious-circle principle" that is beyond all reasonable doubt is simply Russell's law (at least the second-order version). It is impossible that there exists a barber who shaves all and only those villagers, *including him/herself*, who do not shave themselves. Such a barber would indeed be vicious—not vicious like the demon barber of Fleet Street but logically.

Our monadic-predicative proposition *r* (that Socrates is married) is a stone-caster^2. Evidently it is therefore a stone caster *simpliciter*, a stone caster of some level or other. The paradox-producing putative monadic-predicative proposition $\wp$ is not $\wp^{n+1}$ for any $n \geq 1$. (See note 32 above.) Instead $\wp$ is supposed to be the unlevelled proposition that *r* is a stone caster. Ramified type theory blocks the stone-caster paradox by not recognizing unleveled properties, like that of being a stone caster of some level or other, even if only long enough to deny their existence. Perhaps Russell is right that the putative unleveled property of being a stone caster is nonsense. It is in any case not obvious that the putative property itself is the source of the inconsistency. Many propositions seem to lack the property; at least as many seem to possess it. What is certain is that there are no monadic-predicative propositions that predicate the putative property of other monadic-predicative propositions. First-order simple type theory prohibits the postulation of such propositions, by means of Russell's law. To opt for Russell's supplementary prohibition on unleveled propositions or properties is to take a further step.

The paradoxical putative proposition $\wp$ does not exist. There cannot be such a proposition as $\wp$ for the same reason that there is no village barber who shaves all and only those villagers who do not shave themselves: The supposition that there is such a proposition, like the supposition that there is such a barber, is inconsistent, in first-order simple type theory in fact.

Classical first-order logic reveals through Russell's law that there are fewer things in Heav'n and Earth than are dreamt of in much of contemporary philosophy of language. Not only are there no sets of certain kinds; there are also no general terms of certain kinds, no properties of certain kinds, no formulae of certain kinds, no propositions of certain kinds. But not all the news is bad. If you can read this, then you are multi-lingual.

Embrace it. One cannot argue with logic.

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Notes

[1] Russell's paradox was evidently discovered by Ernst Zermelo the year before Russell's independent discovery in the spring of 1901. Zermelo communicated the paradox to David Hilbert, Edmund Husserl, and others. He did not publish it however.

[2] Russell presents the Appendix-B paradox in terms of classes, instead of properties, of propositions. This creates the misimpression that the paradox is set-theoretic. The paradox is also known as *the Russell-Myhill paradox*. See John Myhill, "Problems Arising in the Formalization of Intensional Logic." *Logique et Analyse*, 1 (1958), pp. 78-83.

Throughout the present essay '*q*' is employed as a variable that ranges over propositions.

[3] For a confused alternative solution to Russell's paradox that resembles simple type theory, see Ludwig Wittgenstein, *Tractatus Logico-Philosophicus* (London: Routledge, 1922), 3.333, p. 57.

[4] Alfred Tarski's method of distinguishing languages of different levels is tantamount to stratification of semantic terms (true-in-the-object-language, true-in-the-metalanguage, true-in-the-meta-metalanguage, etc.). *Cf.* Alonzo Church, "Comparison of Russell's Resolution of the Semantical Antinomies with that of Tarski," *Journal of Symbolic Logic*, 41, 4 (1976), pp. 747-760.

[5] A monadic-predicative proposition may be, but need not be, a Russellian singular ("object-involving," or "object-dependent") proposition. A Russellian singular monadic-predicative proposition *directly* predicates a property of an object, whereas the proposition that the teacher of Plato is married (assuming, against Russell, that it is monadic-predicative) *indirectly* predicates being married of Socrates. It is assumed that a monadic-predicative proposition predicates exactly one property, and predicates it of exactly one object.

[6] The pot, itself black, "calls" the kettle black (i.e., accuses the kettle of being black). Only one who is without sin may cast the first stone at a sinner (thereby calling the sinner a sinner).

[7] Here '*x* shaves *y*' is an abbreviation for 'λ*xy*[*x* shaves *y*]'. Similarly throughout.

[8] Russell's law is easily proved by a *reductio ad absurdum* argument—as Russell in effect showed in *The Principles of Mathematics*, chapter X, section 102, pp. 102-103. It is arguable that Russell's law is a variant of a simpler logical truth, one proved by existential generalization: $\forall x \exists y[R(xy) \leftrightarrow R(yy)]$. For every villager (including any village barbers), there is (contrary to the riddle's stipulation) at least one villager such that the former shaves the latter if and only if the latter self-shaves.

James F. Thompson extracts solutions to both Russell's and Grelling's paradoxes from Russell's law. See his "On Some Paradoxes," in R. J. Butler, *ed*., *Analytical Philosophy (First Series)* (Oxford: Basil Blackwell, 1966), pp. 104-119. I learned of Thompson's article only after coming to my own thoughts on Russell's law and the paradoxes. See also Hans Herzberger, "Paradoxes of Grounding in Semantics," *Journal of Philosophy*, LXVII, 6 (March 26, 1970), pp. 145-167; and Robert L. Martin, "On a Puzzling Classical Validity," *Philosophical Review* 86, 4 (1977), pp. 454-473.

[9] I thank Teresa Robertson for providing this reference.

[10] Instantiating the initial universal quantifiers of the second-order version of Russell's law is tantamount to providing both the intended universe for the first-order version and the intended interpretation for the dyadic-predicate constant. Accordingly, the claim that Russell's paradox and a host of similar paradoxes are all partially solved by a single first-order logical law is strictly inaccurate. However, each of the paradoxes in question is partially solved by a particular interpretation instance of Russell's law (and *a fortiori* by the corresponding law of second-order simple type theory), so that each finds a partial solution in first-order logic by means of the same (partially) uninterpreted theorem. For example, the proposition that there is no set whose elements are all and only sets that are not elements of themselves is a truth of first-order logic, a particular instance of Russell's law of first order.

[11] Georg Cantor rejected unrestricted set-theoretic comprehension on the ground that it yields collections that are in one-to-one correspondence with the universe of all sets and hence "too large" to be sets themselves. In "Set Theory with a Universal Set," in L. Henkin, ed., *Proceedings of the Tarski Symposium; Proceedings of Symposia in Pure Mathematics XXV* (Providence, RI: American Mathematical Society, 1974), pp. 297−308, Alonzo Church says, "the principle of limitation of size was never very well supported, and … it should now be abandoned" (p. 297). For further discussion see Michael Hallett, *Cantorian Set Theory and Limitation of Size* (Oxford University Press, 1984).

[12] I thank Harry Deutsch for pointing these remarks out to me.

[13] Given this understanding of the terms (alternatives exist in the literature), it is arguable that there are no proper classes, contrary to the NBG axiom of class formation. I submit that if proper classes are to be countenanced, it is preferable philosophically to take 'set' as primitive, or perhaps defined in terms of a transfinite recursive hierarchy.

[14] This proposal takes 'set' as antecedently understood (see the preceding note 13) and defines a restricted membership relation in terms of it and '$\in$': $x \in y$ & x is a set (or else: $x \in y$ & [x is a set $\vee$ $\sim\exists z(z \in x)$]). The proposal has the immediate consequence that whereas classes other than $\varnothing$ have sets (or *ur*-elements or both) as "members," proper classes are not class "members" *in the restricted, defined sense* (i.e., not class-elements that are themselves sets or *ur*-elements). The proposed interpretation trivializes, and thereby deflates, the otherwise substantive claim that a proper class cannot be an "element" or "member" of any class. On the proposed interpretation, a proper class is never a class "member" not because of girth issues, but because nothing other than a set (or an *ur*-element) counts as a "member" in the relevant sense.

[15] More precisely, the sentence '$\exists x\forall y[\mathrm{R}(xy) \leftrightarrow \mathrm{F}y \,\&\, \sim\mathrm{R}(yy)]$' is satisfiable, but only within the constraint that '$\sim\exists x[\ \mathrm{F}x \,\&\, \forall y(\mathrm{F}y \rightarrow [\mathrm{R}(xy) \leftrightarrow \sim\mathrm{R}(yy)])\ ]$' is a theorem. There could be a village barber who shaves all and only men of the village who don't shave themselves, as long as the barber is not a man, and therefore does not shave herself. Analogously, the proposition that there is a "special" set

whose elements are all and only "run-of-the-mill" sets that are not elements of themselves is formally consistent, whereas Russell's law requires that any such set not be run-of-the-mill (whatever that means). By the same token, the proposition that there is such a class as *R* is formally consistent, while Russell's law requires that it be a proper class (and therefore not an element of itself).

It is arguable that in the intended sense it is analytic that anything with elements is a set. The word 'set' might be defined, for example, so that something is a *set* if and only if it is either ∅ or has elements. This would identify classes with sets. (See notes 13-14 above.) The issues here are complex.

[16] Russell's law is not an existential statement, but a universal. (Strictly, it is a negated-existential statement; see note 8 above.) Nevertheless, both diagonalization arguments using $\mathbf{I_D}$ and $\mathbf{I_{PS}}$ should be regarded as constructive, specifying a missing element through diagonalization. The first specifies for a given enumeration of sequences of 1's and 0's a particular missing sequence of 1's and 0's; the second specifies for a given function from a set into its power set a particular unassigned subset.

[17] In *My Philosophical Development* (New York: Simon and Schuster, 1959), Russell says (pp. 75-76) that he was inspired to conceive of the paradox bearing his name by considering Cantor's proof of a corollary of Cantor's power-set theorem. If **U** is taken to be the universal set and *f* is the identity function, then the specified unassigned subset is Russell's impossible set. (Separation precludes the universal set in ZF set theory.)

[18] I am grateful to Harry Deutsch for discussion of these matters.

[19] If there were such a formula as *True*(*x*), then *Satisfies*(*x*, *y*) would be definable in the language as $\ulcorner True[Subst^{`x'}/_{CD(x)}(y)]\urcorner$, where $Subst^{`x'}/_{CD(x)}(y)$ is a complex open singular term of the language, with variables '*x*' and '*y*' free, that designates the Gödel number of the result of substituting the canonical-numeral designator of *x* for the particular variable '*x*' in the open formula whose Gödel number is *y*.

[20] A proof, correctly attributable to Alan Turing, that no algorithm decides whether an arbitrary program will halt for a given input employs diagonalization. The weakened variant of Gödel's first incompleteness theorem follows also from the undecidability of the halting problem.

In 1999 I attended lectures by Kripke in which he demonstrated that versions and variants of Gödel's first incompleteness theorem, as well as the paradoxes of the Liar and Grelling, may be extracted from the inconsistency of unrestricted set-theoretic comprehension, and hence from Russell's paradox. I had already learned decades earlier that Russell's paradox is partially solved by a first-order logical theorem. I am arguing here that partial solutions to Russell's and Grelling's paradoxes, as well as other results (e.g., the fruit of Cantor's diagonalization) flow directly from Russell's law. By now I am unable to say exactly how my thinking was influenced by Kripke's 1999 lectures. See note 8 above.

[21] According to the *Oxford English Dictionary* the two adjectives are a part of present-day English, so that Grenglish is simply English.

[22] Suppose that semantic terms like 'applies' are stratified, so that a general term applies only *at* a particular level *n* (in the language) to a particular object, or fails to do so. (See note 4 above.) In that case, an adjective of Grenglish is said to be *heterological* if and only if it does not apply *at any level* (does not apply^{n} for any level *n*) in Grenglish to itself.

[23] W. V. Quine proposed such a paradox by means of the verb phrase 'yields a falsehood when appended to its own quotation', in "Ways of Paradox," in his *Ways of Paradox and Other Essays* (New York: Random House, 1966), pp. 3-20, at p. 9.

[24] Robert L. Martin argues (*op. cit*. in note 8 above) that Russell's law is untrue on $\mathbf{I_{het}}$, on the ground (in effect) that a Grenglish formulation would be unintelligible if true. However, Russell's law under $\mathbf{I_{het}}$, if true, has no Grenglish formulation.

[25] Quine makes this point, *op. cit*., p. 6.

[26] Quine says of a principle-schema relevantly like *GT* that "it is a hard principle to distrust, and yet it is obviously the principle that is to blame for our antinomy" (*op. cit*., p. 7).

[27] As Robertson pointed out, other exceptions include such constructions as 'alleged thief' and 'large bug' but these are of an entirely different sort from *h* and its ilk.

[28] Russell's law does not rule out that there is a Grenglish adjective that applies in Grenglish to all and only those English adjectives that do not apply in English to themselves (though this prospect

requires that English ≠ Grenglish). It also does not rule out that there is an English adjective phrase τ that applies in English to all and only those English adjective phrases that that do not apply in English to themselves and are not semantically co-extensional with τ itself. (There are other options.)

[29] See note 22 above. Suppose that a sentence is true or not only at some level *n* (in the language). In that case let the liar sentence be '*L* is not true at any level in English'. The naïve *T*-schema is modified accordingly, replacing 'true' on the left-hand side with 'true at some level or other'.

[30] *American Journal of Mathematics*, 30 (1908), pp. 222-262; reprinted in Russell's *Logic and Knowledge*, ed., Robert C. Marsh (London: George Allen & Unwin Ltd, 1956), pp. 57-102. In *My Philosophical Development* Russell recalls (p. 74; see note 1 above) that Alfred North Whitehead and he began work in 1900 on what was to become *Principia Mathematica*, prior to Russell's discovery of the paradox that bears his name. He also says (p. 79) that he had not found an acceptable solution to the logical paradoxes until 1905. The correct solution, he says, must invoke the theory of types.

[31] See the paper by Church cited in note 4 above. It is stipulated that nothing is heterological^1.

[32] It is stipulated that no proposition is either a kettle-caller^1 or a stone-caster^1.

[33] Simple type theory provides for a more immediate solution to the stone-caster paradox. The construction 'F(F*x*)' is ill-formed because of a clash of simple types. Depending on how the definition of 'stone caster' is formalized, it may be rejected as ill-formed and meaningless on the same ground. The Appendix-B paradox may be treated similarly. However, both paradoxes can be formalized in a manner that remains susceptible to solution by ramified type theory.

[34] Of his classical paradox Russell wrote, "Some other mathematicians [than Henri Poincaré], who disapproved of Georg Cantor, adopted the March Hare's solution: 'I'm tired of this. Let's change the subject.' This, also, appeared to me inadequate" (*My Philosophical Development*, p. 77).

[35] Advocates of supervaluation deny this. However, as a proposal regarding natural-language disjunctions, supervaluation is excessively implausible. Some of my reasons for this judgment are

presented in "Vagaries about Vagueness," in R. Dietz and S. Moruzzi, eds, *Cuts and Clouds* (Oxford University Press, 2010), chapter 7, pp. 131-148, at 140-142.

[36] Not in the sense, however, of *constructing* the object. Although Russell advocated the vicious-circle principle, he was not a mathematical constructivist—as is evidenced, for example, by his adding so-called axioms of reducibility to ramified type theory.

Although popular in some circles, the locution of "defining" a non-linguistic object is very likely the product of confusion. The default province of definitions is neither non-linguistic objects nor concepts but terms, expressions, or most especially, words. To define an expression is to state its semantic meaning in such a way as to convey it to someone who lacks understanding of the expression but not of the definiens. Non-linguistic objects (the moon, my left foot, etc.) typically lack semantic characteristics.